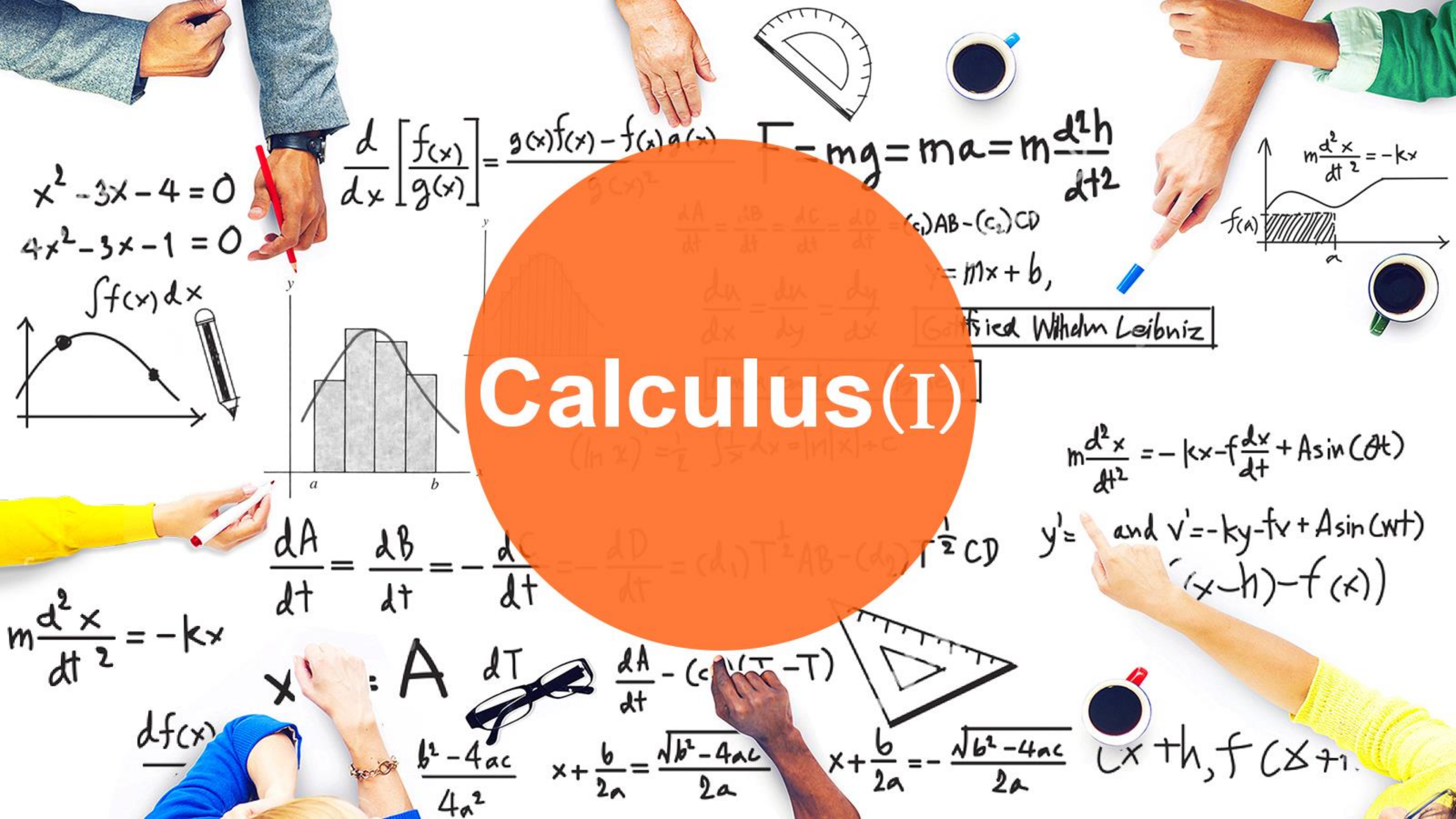
Calculus(I)


x² - 3x - 4 = 0
4x² - 3x - 1 = 0
∫f(x)dx
d/dx [f(x)/g(x)] =
= mg = ma = m d²h/dt²
m d²x/dt² = -kx
f(a)
a
y = mx + b,
Gottfried Wilhelm Leibniz
m d²x/dt² = -kx - f dx/dt + Asin(θt)
and v' = -ky - fv + Asin(wt)
dA/dt = dB/dt = - dC/dt
m d²x/dt² = -kx
df(x)
b² - 4ac / 4a²
x + b/2a = √b² - 4ac / 2a
x + b/2a = - √b² - 4ac / 2a

Limits at Infinity; Infinite Limits

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Problem Introduction

How to find $\lim_{x \to +\infty} f(x)$, $\lim_{x \to -\infty} f(x)$ and $\lim_{x \to \infty} f(x)$?

- $x \to +\infty$ — x tends to positive infinity
- $x \to -\infty$ — x tends to negative infinity
- $x \to \infty$ — x tends to infinity

Definition Limit as $x \to +\infty$

Let f be defined on $[c, +\infty)$ for some number c. We say that $\lim\limits_{x\to+\infty} f(x) = L$ if for each $\varepsilon > 0$ there is a corresponding number M such that

$$x > M \Rightarrow \left| f(x) - L \right| < \varepsilon$$

Definition Limit as $x \to -\infty$

Let f be defined on $(-\infty, c]$ for some number c. We say that $\lim_{x \to -\infty} f(x) = L$ if for each $\varepsilon > 0$ there is a corresponding number M such that

$$x < M \Rightarrow |f(x) - L| < \varepsilon.$$

Definition Limit as $x \to \infty$

Let f be defined on $(-\infty, +\infty)$. We say that $\lim_{x \to \infty} f(x) = L$

if for each $\varepsilon > 0$ there is a corresponding number M such that

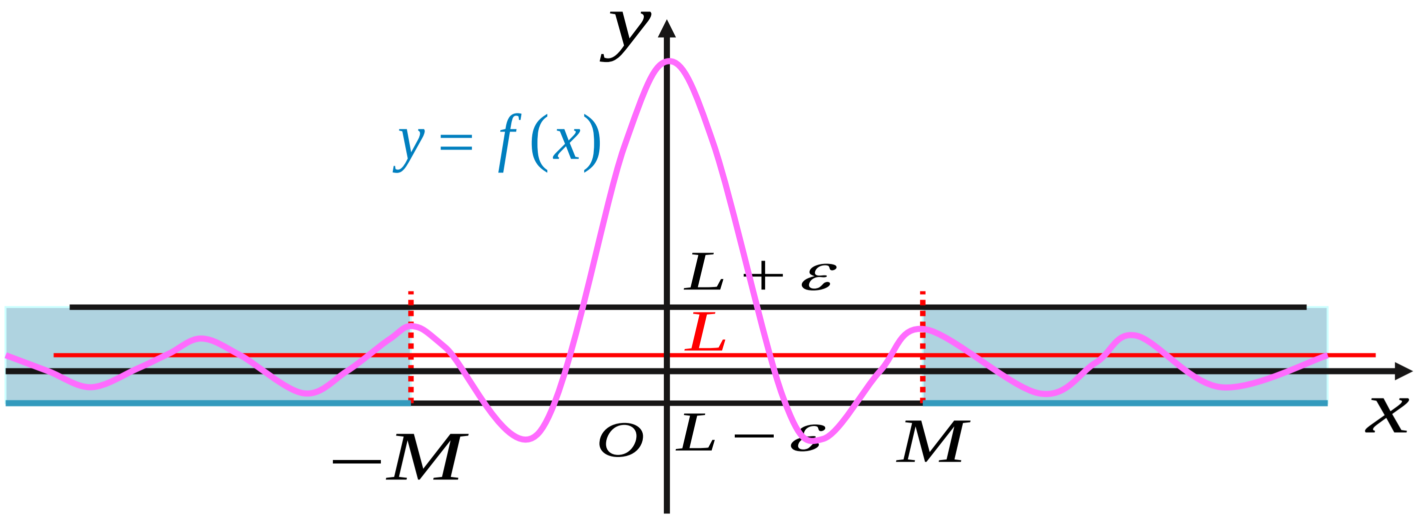
$$|x| > M \Rightarrow |f(x) - L| < \varepsilon$$

Note: $\lim_{x \to \infty} f(x) = L \Leftrightarrow \lim_{x \to +\infty} f(x) = L = \lim_{x \to -\infty} f(x)$

Geometrical meaning of $\lim_{x\to\infty} f(x) = L$

$\forall \varepsilon > 0$, $\exists M > 0$, when $|x| > M$, we have

$$|f(x) - L| < \varepsilon \iff L - \varepsilon < f(x) < L + \varepsilon$$



Definition Limit of a Sequence

Let a_n be defined for all natural numbers greater than or equal to some number c. We say that $\lim_{n\to\infty} a_n = L$ if for each $\varepsilon > 0$ there is a corresponding natural number M such that

$$n > M \Rightarrow |a_n - L| < \varepsilon.$$

Definition Infinite Limits

We say that $\lim_{x \to c^+} f(x) = +\infty$ if for every positive number M, there exists a corresponding $\delta > 0$ such that

$$0 < x - c < \delta \Rightarrow f(x) > M$$

Example 1

Find $\lim_{x\to\infty}\frac{3x^2+2x-1}{x^3-3x+5}$. $\left(\frac{\infty}{\infty}\right)$

$$=\lim_{x\to\infty}\frac{\frac{3}{x}+\frac{2}{x^2}-\frac{1}{x^3}}{1-\frac{3}{x^2}+\frac{5}{x^3}}=\frac{0}{1}=0.$$

Note: ($a_0\neq 0, b_0\neq 0, m, n$ are nonnegative integers)

$$\lim_{x\to\infty}\frac{a_0x^m+a_1x^{m-1}+\cdots+a_m}{b_0x^n+b_1x^{n-1}+\cdots+b_n}=\begin{cases}\frac{a_0}{b_0} & n=m\\ 0 & n>m\\ \infty & n<m\end{cases}$$

Example 2

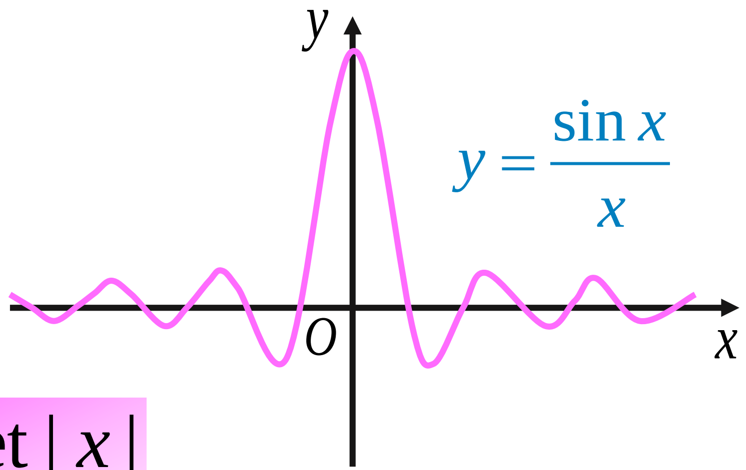
Find $\lim\limits_{x\to\infty}\dfrac{x^3-2x+5}{3x^5+2x+3}(2+\cos x-3\sin x)$.

$$\because \lim_{x\to\infty}\frac{x^3-2x+5}{3x^5+2x+3}=0, \quad |2+\cos x-3\sin x|\le 6$$

$$\therefore \lim_{x\to\infty}\frac{x^3-2x+5}{3x^5+2x+3}(2+\cos x-3\sin x)= \mathbf{0}\ .$$

Example 3

Prove $\lim\limits_{x\to\infty}\dfrac{\sin x}{x}=0.$



For $\forall\varepsilon>0$, in order to $\left|\dfrac{\sin x}{x}-0\right|<\varepsilon$

$\because \left|\dfrac{\sin x}{x}-0\right|=\left|\dfrac{\sin x}{x}\right|\le\dfrac{1}{|x|}$, just need $\dfrac{1}{|x|}<\varepsilon$ get $|x|$

namely $|x|>\dfrac{1}{\varepsilon}$, let $M=\dfrac{1}{\varepsilon}$, when $|x|>M$,

we have $\left|\dfrac{\sin x}{x}-0\right|<\varepsilon$, then $\lim\limits_{x\to\infty}\dfrac{\sin x}{x}=0.$

Example 4

Prove that $\lim_{n\to\infty}\dfrac{n+(-1)^{n-1}}{n}=1.$

$$\forall \varepsilon>0,\quad |a_n-1|=\left|\frac{n+(-1)^{n-1}}{n}-1\right|=\left|\frac{(-1)^{n-1}}{n}\right|=\frac{1}{n}$$

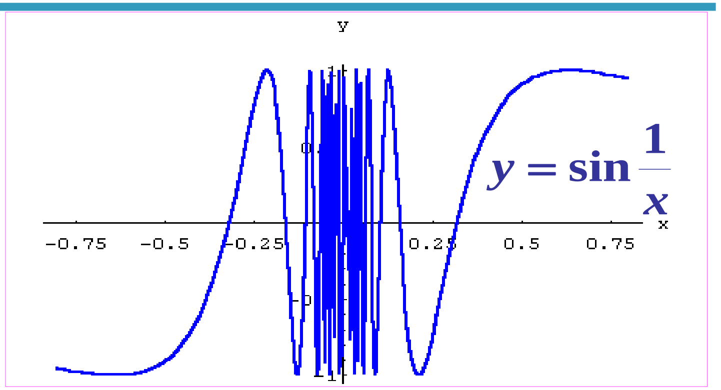
let $|a_n-1|<\varepsilon$, we have $\dfrac{1}{n}<\varepsilon$ or $n>\dfrac{1}{\varepsilon}$

choose $M=\left[\dfrac{1}{\varepsilon}\right]$, when $n>M$,

we have $\left|\dfrac{n+(-1)^{n-1}}{n}-1\right|<\varepsilon$ namely, $\lim_{n\to\infty}\dfrac{n+(-1)^{n-1}}{n}=1.$

Example 5

Prove $\lim_{x\to 0}\sin\frac{1}{x}$ doesn't exist.



Let $\{x_n\}=\left\{\frac{1}{n\pi}\right\}$, so $\lim_{n\to\infty}\sin\frac{1}{x_n}=\lim_{n\to\infty}\sin n\pi=0$,

Let $\{x'_n\}=\left\{\frac{1}{\frac{4n+1}{2}\pi}\right\}$, then $\lim_{n\to\infty}\sin\frac{1}{x'_n}=\lim_{n\to\infty}\sin\frac{4n+1}{2}\pi=\lim_{n\to\infty}1=1$,

The two limits are not the same, so $\lim_{x\to 0}\sin\frac{1}{x}$ doesn't exist.

$x_n\to 0, x'_n\to 0 \quad (n\to\infty)$

Summary of Limits at Infinity and Infinite Limits

1. $\lim\limits_{x\to+\infty} f(x)=L \Leftrightarrow \forall\varepsilon>0, \exists M>0, \text{s.t. } x>M \Rightarrow |f(x)-L|<\varepsilon.$

2. $\lim\limits_{x\to-\infty} f(x)=L \Leftrightarrow \forall\varepsilon>0, \exists M>0, \text{s.t. } x<M \Rightarrow |f(x)-L|<\varepsilon.$

3. $\lim\limits_{x\to\infty} f(x)=L \Leftrightarrow \forall\varepsilon>0,\ \exists M>0,\ \text{s.t. } |x|>M \Rightarrow |f(x)-L|<\varepsilon.$

4. $\lim\limits_{n\to\infty} a_n=L \Leftrightarrow \forall\varepsilon>0,\ \exists M>0,\ \text{s.t. } n>M \Rightarrow |a_n-L|<\varepsilon.$

Questions and Answers

Q1: Find $\lim_{n\to\infty}\left[\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2n-1)(2n+1)}\right]$.

$$\lim_{n\to\infty}\frac{1}{2}\left[1-\frac{1}{3}+\frac{1}{3}-\frac{1}{5}+\cdots+\frac{1}{2n-1}-\frac{1}{2n+1}\right]$$

$$=\lim_{n\to\infty}\frac{1}{2}\left(1-\frac{1}{2n+1}\right)$$

$$=\frac{1}{2}.$$

Questions and Answers

Q2: Find $\lim_{x\to 2}\frac{x+1}{x^2-5x+6}$.

(1) $\lim_{x\to 2^+}\frac{x+1}{(x-2)(x-3)}=-\infty$ $\left(\frac{+}{+\cdot -}\right)$

(2) $\lim_{x\to 2^-}\frac{x+1}{(x-2)(x-3)}=+\infty$ $\left(\frac{+}{-\cdot -}\right)$

$\therefore$ The limit does not exist!

Questions and Answers

Q3: Prove that $\lim\limits_{n\to\infty}\frac{3n-1}{n^2+n+1}=0.$

$\forall\varepsilon>0, \exists M=\left[\frac{3}{\varepsilon}\right]$ such that $n>M$, then

$$\left|\frac{3n-1}{n^2+n+1}-0\right|=\frac{3n-1}{n^2+n+1}<\frac{3n}{n^2}=\frac{3}{n}<\varepsilon.$$

namely, $\lim\limits_{n\to\infty}\frac{3n-1}{n^2+n+1}=0.$

Questions and Answers

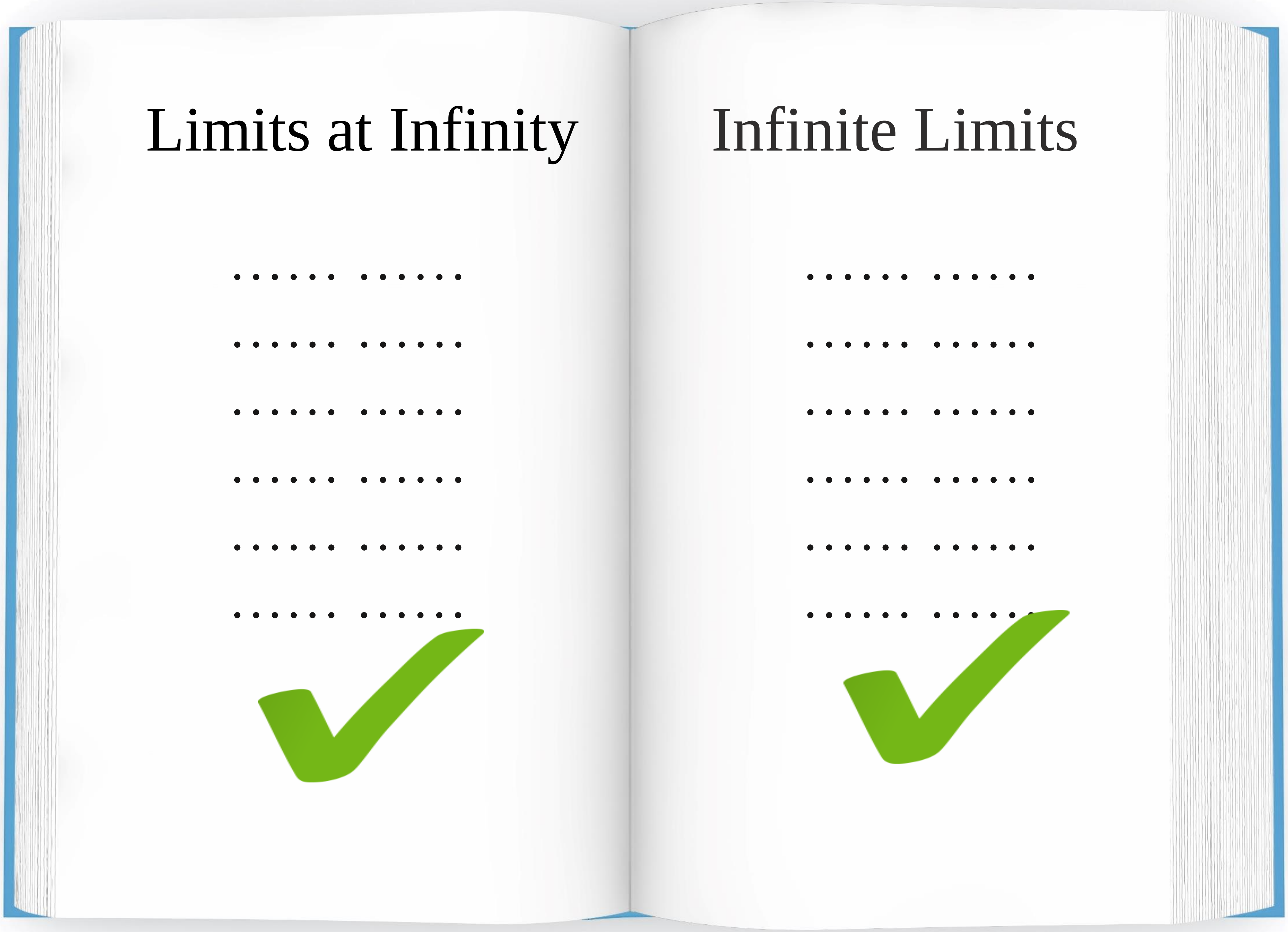
Q4: Prove $\lim\limits_{x\to+\infty}\dfrac{x^2-1}{x^2+1}=1.$

when $x>0$, we have $\left|\dfrac{x^2-1}{x^2+1}-1\right|=\dfrac{2}{x^2+1}<\dfrac{2}{x^2}$,

$\forall\varepsilon>0$, in order to $\left|\dfrac{x^2-1}{x^2+1}-1\right|<\varepsilon$, need $\dfrac{2}{x^2}<\varepsilon$, get x

$x>\sqrt{\dfrac{2}{\varepsilon}}$, let $M=\sqrt{\dfrac{2}{\varepsilon}}$, when $x>M$, we have

$$\left|\frac{x^2-1}{x^2+1}-1\right|<\frac{2}{x^2}<\varepsilon \quad \therefore \lim_{x\to+\infty}\frac{x^2-1}{x^2+1}=1.$$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Infinite Limits